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We show that appropriate union structures will generate a non-empty core for any game which is monotonic in zero-normalised form (such as a superadditive game). We give simple numerical examples of games which require more than one union and indeed more than one "level" of unions to generate a non-empty core. Finally we characterise the complexity of the union structure required for different sizes of games, in particular for symmetric games. We conclude with some remarks on which union structures are likely to form.

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ON THE FORMATION OF UNIONS IN n-PERSON GAMES

by

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ABSTRACT

A union in an n-person cooperative game is a group of players who agree to act only in unison. Unions may combine to form further unions, and sub-unions may form within a union thereby generating a union structure. The effect on the core of the game is, if anything, to enlarge it by removing potential blocking coalitions.

We show that appropriate union structures will generate a non-empty core for any game which is monotonic in zero-normalised form (such as a superadditive game). We give simple numerical examples of games which require more than one union and indeed more than one "level" of unions to generate a non-empty core. Fianlly we characterise the complexity of the union structure required for different sizes of games, in particular for symmetric games. We conclude with some remarks on which union structures are likely to form.

On The Formation of Unions in n-Person Games†

I INTRODUCTION

In n-person cooperative game theory one may distinguish between potential coalitions and coalitions which actually form. We are concerned here with the latter. We define a union as a group of players who make a binding agreement to act only in unison in order to increase their bargaining power. Unions may combine to form further unions, and sub-unions may also form within a union, thereby building up a union structure with several levels (Section III). The effect of this procedure is, if anything, to enlarge the core by removing potential blocking coalitions (Section IV).

Is it always possible to generate a non-empty core by forming such unions? In this paper we show that an appropriate union structure will generate a non-empty core for any game which is monotonic in zero-normalised form, in particular for super additive games. (Section V) However, this is not usually a trivial

† This work was prompted by numerous discussions with S. Sorensen concerning his earlier work with Charnes on hierarchical games and core-stem solutions: we are grateful to him for extensive written comments on previous drafts of this paper. We have also benefited from discussions with J. Dreze, A. Postlewaite, M. Sertel and L. Wolsey. In particular the analysis of sections VII and VIII was developed in response to a question raised by Postlewaite. The initial research by Littlechild was partially supported by the Mathematics Department of General Motors Research Laboratory in October 1971.

matter. Indeed, we give simple numerical examples of quite small games which require more than one union, and indeed more than one "level" of unions, to generate a non-empty core. We also compute upper bounds on the number of unions and levels of unions required as a function of the number of players (Section VI). For symmetric games lower and more precise bounds are obtained (Section VII). For example, we show that an arbitrary symmetric superadditive 1000-person game may be reduced to a game between at most 9 unions such that each "union subgame" has a non empty core. If the core of the 9-person is empty then a non-empty core for the original game may be generated by forming at most 7 more unions at 3 more levels. We conclude with some remarks about how the bargaining procedure between and within unions is likely to affect which union structures form (Section VIII).

Several previous authors have examined the effect of coalition structures in characteristic function games. Their methods and intentions have varied greatly and it is not easy to compare the different approaches. With the exception of Postlewaite and Rosenthal [12] these authors have not been concerned to relate non-emptiness of the core

to size of game and complexity of coalition structure. In the present paper we define a union as Jaskold-Gabszewicz and colleagues [7, 8, 9] and Postlewaite and Rosenthal define a syndicate, but unlike these authors, we allow several levels of unions to form. The approach of Aumann and Maschler [3] and Aumann and Dreze [2] is also limited to one level. The coalition structures of Nering [11] and Radstrom [13] have several levels, and correspond in some ways to special cases of our own union structures (which we call ultimate union structures) but their own coalition structures have additional restrictions. Our approach is thus more general than most of these papers. In some respects it is an alternative characterisation of the "hierarchical games" and "core-stem solutions" introduced by Charnes and Sorenson [4, 5, 6, 14] which indeed stimulated the present paper.

Techniques and notation in previous papers have been quite complex and not easily accessible to the non-specialist reader. The present paper is relatively elementary, with

graphical illustrations and numerical examples to aid intuition. It will be convenient to begin with three simple examples.

II SIMPLE EXAMPLES

EXAMPLE 1

Consider the following situation[1]. A lady offers one dollar to have her trunk carried. A big man and two small men are available, none of whom can carry the trunk alone, nor can the two small men together carry the trunk, but the big man and either of the two small men can carry it. How should the men divide up the dollar?

If we denote the big man by player 1 and the two small men by players 2 and 3, the situation can be represented as a three-person game with characteristic function

$$v(\phi) = v(1) = v(2) = v(3) = v(2,3) = 0$$

$$v(1,2) = v(1,3) = v(1,2,3) = 1 .$$

The core of this game is the single imputation (1,0,0) ie, one dollar to the big man and nothing to either of the small men.

This is in accord with the interpretation of the core as the set of competitive outcomes: since only one small man is needed to carry the trunk then the marginal productivity of a second small man is zero. If one small man were to be offered any positive reward then the other would be willing to work for less, and so their wage would be bargained down to zero.

If this were the only solution, it would be a rather depressing world for small men to live in. But in practice small men do have a remedy available to them which they readily take advantage of, namely they form a union to bargain for them. Suppose, then, that players 2 and 3 form a union to bargain for them which we denote player 4. The new situation can be represented as a two-person game with characteristic function

$$w(\phi) = w(1) = w(4) = 0, \; w(1,4) = 1.$$

Now the situation is quite different. The core of this game is the set of all possible divisions of the dollar between the big man and the union. In principle the amount gained by the union could be split in any way between its two members (although, being identical, they might well decide to share it

equally). Hence the final outcome might be any possible division of the dollar btween the three participants. In a sense, the formation of the union widened the core of the original game from the single imputation (1,0,0) to the whole 3-dimensional unit simplex.

Example 2

Consider the same situation but with three big men only rather than one big man and two small men.[2] The characteristic function is now

$$v(\phi) = v(1) = v(2) = v(3) = 0$$

$$v(1,2) = v(1,3) = v(2,3) = v(1,2,3) = 1.$$

Since the game is symmetric, if there exists an imputation in the core then there must exist a symmetric imputation in the core. The only symmetric imputation $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ can be improved upon by any pair of players, hence the core is empty.

One might expect that in this situation each player would be anxious to pair up with either of the others. Suppose players 2 and 3 form a union, denoted player 4. The characteristic function of the resulting two-person game is

$$w(\phi) = w(1) = 0, \quad w(4) = w(1,4) = 1.$$

The core of this game consists of the single imputation (0,1), ie. nothing to the player left out and one dollar to the union.

Assuming the union can divide up its pay-off in any way, the core of the original game is the set of points (0,a,1-a) for $0 \leq a \leq 1$. Thus a non-empty core has effectively been generated for the original game.

Example 3

Consider the symmetric 4-person game given by

$$v(S) = \begin{cases} 0, & |S| = 0,1 \\ 1, & |S| = 2,3 \\ 2, & |S| = 4 \end{cases}$$

where $|S|$ denotes the number of players in coalition S. The unique point in the core is $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$. If the first three players form a union, denoted player 5, the resulting game is given by

$$w(4) = 0, \quad w(5) = 1, \quad w(4,5) = 2.$$

The union may therefore get anything between 1 and 2 - but if its pay-off is less than $1\frac{1}{2}$ it will not be able to satisfy its members, because the core of the "union sub game" is non-empty.

For this reason it would be risky to form such a union; for the most part we shall restrict our attentions to "proper" unions whose "subgames" have non-empty cores, so that this difficulty does not arise. This does not mean that a proper union always improves the lot of its members. Indeed, Postlewaite and Rosenthal [12] , following Aumann [1], have produced a

4-person game where all the new points in the core generated by a proper union are disadvantageous to the members of the union.

III CONCEPTS AND DEFINITIONS

A game in characteristic function form is an ordered pair (N,v) where $N = \{1,\ldots,n\}$ is a finite set of players and v, called the characteristic function, is a non-negative real-valued function defined on every subset of N and satisfying $v(\phi) = 0$. A payoff vector for (N,v) is a point x in $|N|$-dimensional vector space E^N whose coordinates x_i are indexed by the elements of N. If $x \in E^N$ and $S \subseteq N$ we write $x(S)$ as an abbreviation for $\sum_{i \in S} x_i$. $S \subset N$ will denote $S \subseteq N$ but $S \neq N$. The set of imputations for (N,v) is the set M of payoff vectors which allocate the value of the "grand coalition" N and which cannot be improved upon by any individual player, or

(1) $M = \{x : x_i \geq v(\{i\}),\ i \in N, \text{ and } x(N) = v(N)\}$.

The core of (N,v) is the set C of payoff vectors which allocate the value of the "grand coalition" and which cannot be improved upon by any subset of players, or

(2) $C = \{x : x(S) \geq v(S),\ S \subset N, \text{ and } x(N) = v(N)\}$.

With these definitions the set of imputations and the core always exist and are convex but may be empty. Evidently the core is contained in the set of imputations. A game with a non-empty core is said to be proper.

A <u>union</u> S on a set of players N is a subset of players $S \subseteq N$ who agree that no proper subset of them will form a coalition with players outside S. However, players may form coalitions within the union and the union as a whole may enter into broader coalitions. If S and T are unions and $S \subset T$ then S is called a <u>sub-union</u> of T. At any time, therefore, there will be a <u>union structure</u> $\mathcal{U}$ on a set of players N. This is defined as a collection of unions $S \subseteq N$ such that if S_1 and S_2 are in $\mathcal{U}$ then $S_1 \cap S_2 \neq \emptyset$ implies $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. Thus if two unions intersect, one must be a sub-union of the other. The sets $\emptyset$, (1), (2), ... (n) and N are called <u>inessential</u> unions, and are assumed to be included in every union structure. In fact, the union structure of the original game, denoted $\mathcal{U}_0$, consists only of the inessential unions. Evidently a union structure is closed under set-theoretic intersection, that is

$$S_1, S_2 \in \mathcal{U} \rightarrow S_1 \cap S_2 \in \mathcal{U}$$

but it is not closed under set-theoretic union. {It therefore constitutes a semi-lattice with respect to set inclusion} ¶ We

shall say that the formation of further unions refines the union structure. Formally, if $\mathcal{U}_1$ and $\mathcal{U}_2$ are two union structures defined on N then $\mathcal{U}_2$ is said to be strictly finer than $\mathcal{U}_1$ if $\mathcal{U}_1 \subset \mathcal{U}_2$. (The property of containment for union structures is defined in the usual way, viz. $\mathcal{U}_1 \subseteq \mathcal{U}_2$ if $S \in \mathcal{U}_1 \rightarrow S \in \mathcal{U}_2$ and $\mathcal{U}_1 \subset \mathcal{U}_2$ if $\mathcal{U}_1 \subseteq \mathcal{U}_2$ but not $\mathcal{U}_1 = \mathcal{U}_2$.) A union structure may be represented visually in terms of "contour lines" as in Figure 1, where the dots represent players and the contour lines unions. A union structure has the property that the contour lines never intersect.

Note that the union structure contains no information about the sequence of formation of unions. It would not be recorded whether players A and B form a union and subsequently add player C, or whether players A, B and C first form a union and subsequently players A and B form a sub-union. Rather, the union structure provides a "snapshot" at a moment in time. This considerable simplification is available because we do not assume that rules for the division of the payoff are laid

down upon the formation of a union. Instead, we are interested in characterising the possible range of payoffs or core generated by a given union structure. This point is discussed further in the final section.

IV CORES GENERATED BY UNION STRUCTURES

We may think of a union structure as defining a set of associated "sub-games", one for each essential union, plus a "maximal subgame" for the inessential union N. The players in each union subgame are themselves unions, specifically the largest sub-unions contained in that union. The charac-teristic function of the subgame is the original characteristic function restricted to the appropriate set of union players. Formally, for each essential union $S \in \mathcal{U}$, define the <u>union subgame</u> $(\tilde{S}, v_{\tilde{S}})$. Here $\tilde{S}$ denotes the set of union players consisting of all those unions in S which are not contained in any other union in S. Thus T is a union-player in subgame $(\tilde{S}, v_{\tilde{S}})$ if $T \in \mathcal{U}$, $T \subset S$ and there is no union $R \in \mathcal{U}$ such that $T \subset R \subset S$. $v_{\tilde{S}}$ is the original charac-teristic function v restricted to this set of union-players. Thus if T_1,

T_2, ..., are union-players in $(\tilde{S}, v_{\tilde{S}})$ then $v_{\tilde{S}}(\{T_1, T_2, \ldots\})$ $= v(T_1 \cup T_2 \cup \ldots)$. The <u>maximal subgame</u> $(\tilde{N}, v_{\tilde{N}})$ is defined analogously. A union structure is said to be <u>individually proper</u> if all the union subgames, including the maximal subgame, are proper (that is, have non-empty cores). In each of the three examples, the maximal subgame has two players, with characteristic function w, and there is one other union subgame. Examples 1 and 2 have individually proper union structures but example 3 does not.

Suppose the largest unions bargain between themselves in the maximal subgame. The payoff to each union is then bargained for by its constituent unions in its own subgame, and so on down to the original individual players. We therefore have a collection of <u>modified subgames</u> where the value of the "grand coalition" S in any subgame $(\tilde{S}, v_{\tilde{S}})$ is taken to be, not the characteristic function value v(S), but rather the payoff to the union S as union-player in a previous (or higher) subgame. Suppose the payoff vector is in the core of the higher subgame, so that the payoff to union-player S exceeds v(S). If the subgame $(\tilde{S}, v_{\tilde{S}})$ is proper

then so will be the corresponding modified subgame. However, if the subgame $(\tilde{S}, v_{\tilde{S}})$ is not proper the corresponding modified subgame might or might not be proper. Thus, in example 3, the modified subgame is proper if and only if the payoff to the union is not less than 1½.

More generally, if a union structure is individually proper, then all the modified subgames will be proper. In this case it will be possible to divide up the payoffs so that at each stage the payoff vectors lie in the cores of the modified subgames. We shall define the core generated by a union structure as the set of payoff vectors for the original players which satisfy this procedure.

An explicit expression for the core may be obtained as follows. A payoff vector is in the core if it is not blocked by any coalition of union-players in any subgame. A set of original players forms a blocking coalition in some subgame if and only if it either contains or is contained in every union in the union structure with which it has a non-empty intersection. Put more simply, proper

subsets of members of one union may not form blocking coalitions with players outside that union; the union negotiates with outsiders as a body. Formally, the <u>allowable set of blocking coalitions (for the whole game) under union structure</u> $\mathcal{U}$, denoted by $\mathcal{B}(\mathcal{U})$, is given by

$$\mathcal{B}(\mathcal{U}) = \{S \subseteq N: \text{ if } S \cap S' \neq \emptyset \text{ then } S \subseteq S' \text{ or } S \supseteq S' \text{ for all } S' \in \mathcal{U}\}.$$

In examples 1 and 2, coalitions $\{1, 2\}$ and $\{1, 3\}$ are no longer allowable blocking coalitions. However, the set of blocking coalitions $\mathcal{B}(\mathcal{U})$ evidently contains all the unions in the union structure $\mathcal{U}$ itself, and in general more besides. Only the set of blocking coalitions $\mathcal{B}(\mathcal{U}_0)$ for the original union structure $\mathcal{U}_0$ contains all subsets of N.

We now define the <u>core generated by union structure</u> $\mathcal{U}$, denoted $C(\mathcal{U})$, as the set of payoff vectors which cannot be improved upon by any blocking coalition in the allowable set, or

$$C(\mathcal{U}) = \{x: x(S) \geq v(S),\ S \in \mathcal{B}(U), \text{ and } x(N) = v(N)\}.$$

This definition of the core corresponds to that of Gabszewicz and Dreze [8].

Certain elementary properties of the core follow immediately. It is convex because it is refined by a set of linear equalities and inequalities. It contains the core because certain blocking coalitions have been removed, and it lies in the set of imputations because individual players (inessential unions) can always block. Refining the union structure by forming further unions widens the core in the weak sense of not making it smaller, but does not necessarily enlarge it or generate a non-empty core from an empty one. These properties are summarised in

Proposition 1

(a) $C(\mathcal{U})$ is convex

(b) $C \subseteq C(\mathcal{U}) \subseteq M$

(c) $\mathcal{U}_1 \subset \mathcal{U}_2$ implies $C(\mathcal{U}_1) \subseteq C(\mathcal{U}_2)$.

If the core generated by a union structure is non-empty, we shall say that the union structure is <u>proper</u>. As remarked earlier, if the union structure is individually proper we can always generate at least one point in the core, thereby establishing

Lemma 1 An individually proper union structure is proper.

However, as example 3 shows, the converse of lemma 1 is

not true. The payoff vector (5/8, 5/8, 5/8, 1/8) is in the core because payoff 15/8 in the maximal subgame makes the modified union subgame proper, but the union subgame itself is not proper, consequently the union structure is not individually proper.

In what follows we shall restrict ourselves to proper union subgames, where the non-emptiness of the core does not depend upon the payoff in a higher subgame. Our general strategy will be to form new unions until the union structure is individually proper, in which case non-emptiness of the core is guaranteed.

V NON-EMPTINESS OF THE CORE

In the present section we shall show that, by appropriate choice of union structure, a non-empty core may be generated for all super-additive games and for all games which are monotonic in zero-normalized form. A function v on N is <u>super-additive</u> if and only if $v(S_1 \cup S_2) \geq v(S_1) + v(S_2)$ for all non-intersecting S_1, S_2 in N, and <u>monotonic</u> if $v(S_1) \leq v(S_2)$ for all S_1 contained in S_2 in N.

It is in <u>zero-normalized form</u> if $v(\{i\}) = 0$ for all i in N. Recall that v is non-negative.

Define an <u>ultimate union structure</u>, denoted $\mathcal{U}^*$, as a union structure such that there is no other union structure on N which is strictly finer than it (see Fig 2). Evidently for 3 or more person games it is not unique. In fact, Radstrom [13] has shown that there are 1.3.5...(2n-3) different ultimate union structures on a set of $n \geq 2$ players. An ultimate union structure has the property that the unions themselves are the only allowable blocking coalitions, that is $\mathcal{B}(\mathcal{U}^*) = \mathcal{U}^*$. Moreover, since there is no scope for more unions to form, every 2-or-more-person union may be decomposed into exactly two constituent unions, otherwise there would be scope to form more unions. Formally

<u>Lemma 2</u> For all unions $S \in \mathcal{U}^*$ with $|S| > 1$ there exist mutually disjoint and non-null unions $S_1, S_2 \in \mathcal{U}^*$ such that $S = S_1 \cup S_2$.

Say that a union structure is <u>centred</u> if at least one original player is contained in all essential unions in $\mathcal{U}$

(see Fig 3). In particular, if an ultimate union structure is formed by starting with one player and building a sequence of unions by successively adding one player at a time, then the resulting structure will be centred, and each 2-or-more-person union will contain a 1-person union and one other union (see Fig 4).

Proposition 2

(a) If a game is superadditive then the core generated by any ultimate union structure is non-empty.

(b) If a game is monotonic in zero-normalised form then the core generated by any centred ultimate union structure is non-empty.

Proof

In either case all subgames are 2-person, by lemma 2. They are also superadditive, in case (a) by assumption and in case (b) because any 2-person monotonic game with zero value for one of the players is superadditive. But all 2-person superadditive games are proper, hence in both cases the union structures are individually proper, and the result follows from lemma 1. □

It should be noted that even for a super-additive game the core cannot in general be widened to comprise the whole of the set of imputations. In fact, there may be some imputations which cannot be included in any generated core. In example 2, it is possible to generate a core consisting of any one extreme point of M, but not containing any interior points.

VI SIZES OF UNION STRUCTURES

We have shown that, by the formation of appropriate ultimate union structures, a non-empty core may be generated for all games which are monotonic in zero-normalized form. We shall show that an ultimate union structure always has n-2 unions and may have up to n-2 different "levels". In order that the present approach be of interest, we need to show on the one hand that it is not possible to generate empty cores for all superadditive and monotonic games by forming a single union, or even a single level of union, and on the other hand that for significant classes of such games it is not necessary to form anything as complicated as an ultimate union structure. More generally, it may be useful to classify

games according to the difficulty of generating an empty core. In the present section we provide a few simple examples to establish the first point; in the next section we provide a classification of symmetric superadditive games which suggests that considerably fewer unions will be required than the upper bound of n-2.

<u>Proposition 3</u>

An ultimate union structure on an n-person game contains n-2 unions (for $n \geq 2$).

<u>Proof</u>

The proof is by induction on n. Evidently the proposition holds for n=2. Suppose it holds for arbitrary $n > 2$. Consider an ultimate union structure on an n+1-person game. If any one player is deleted then the smallest union containing the deleted player now contains the same set of players as the largest union not containing the deleted player. One of these two unions is therefore redundant and may be deleted to leave an ultimate union structure on an n-person game. By assumption this has n-2 unions so the n+1-person game had n-1 unions.□

Say that the number of levels in a union structure is the maximum number of essential unions in which any player is contained. Define a bisecting ultimate union structure as one in which the number of original players in the two sub-unions contained in any union differs by at most one player. The proof of the following proposition is elementary but tedious, and will be omitted.

Proposition 4 For an n-person game

(a) a centred ultimate union structure contains n-2 levels;

(b) a bisecting ultimate union structure contains h levels, where $2^h < n \leq 2^{h+1}$;

(c) the number of levels in any ultimate union structure lies between these two extremes.

To illustrate the results obtained so far, we have shown that a non -empty core could be generated for an arbitrary 1000-person superadditive game by the formation of an ultimate union structure (proposition 2) involving at most 998 unions (proposition 3) on at most 9 levels (proposition 4b). The next section will show that,

for symmetric games at least, it will not be necessary to form anything like as many unions. We conclude this section with three examples to show that even in small games more than one union and more than one level of unions may be necessary to generate an empty core. The examples are all symmetric games since these are easier to work with, but non-symmetric examples can easily be generated by slightly perturbing the games. We shall restrict ourselves to individually proper union structures.

Example 4

$$v(S) = \begin{cases} 0, & |S| = 0,1 \\ 1, & |S| = 2,3,\ldots,n \end{cases}$$

This is a monotonic game which requires an ultimate centred union structure to generate a non-empty core, hence n-2 unions at n-2 levels.

For superadditive games, example 2 was a 3-person game requiring one union, example 5 is a 5-person game requiring two unions and example 6 is a 7-person game requiring two levels of unions. In what follows we shall use the term 'm-person subgame' as an abbreviation for 'subgame corresponding to an m-person union'.

Example 5

$$v(S) = \begin{cases} 0, & |S| = 0,1 \\ 1+\varepsilon, & |S| = 2 \\ 2, & |S| = 3 \\ 2+2\varepsilon, & |S| = 4 \\ 3+\varepsilon, & |S| = 5 \end{cases}$$

It may be verified that for $\varepsilon=0$, any 5- or 4-person subgame would not be proper but a 3-person subgame would be. Moreover, forming a 3-person union generates a proper 3-person maximal subgame. However, for $0<\varepsilon<\frac{1}{3}$ this maximal subgame is not proper and it is necessary to form a further union of the two remaining players (or else move to a second level by adding one player to the 3-person union).

Example 6

$$v(S) = \begin{cases} 0, & |S| = 0,1 \\ 1, & |S| = 2 \\ 1+\varepsilon & |S| = 3 \\ 3, & |S| = 4 \\ 3+\varepsilon & |S| = 5 \\ 4+\varepsilon & |S| = 6,7 \end{cases}$$

For $\epsilon=0$, the 7-,6-,5- and 3-person subgames are not proper. If a 4-person union forms, the resulting 4-person maximal subgame is not yet proper but if a 2-person union forms as well then the resulting 3-person maximal subgame is proper. However, for $0 < \epsilon < \frac{1}{2}$ this maximal subgame is not proper and it is necessary to form a second level of unions.

VII SYMMETRIC GAMES

This section will show that less complex union structures are required to generate non-empty cores for symmetric super-additive games. The following lemma will be useful. Let $v(\bar{k})$ denote the characteristic function value of a coalition of k players.

<u>Lemma 3</u>

A symmetric n-person game is proper if and only if

$v(\bar{n}) \geq \frac{n}{k} v(\bar{k})$ for $k=1,2,\ldots,n-1$.

<u>Proof</u>

If the core of a symmetric game is non-empty it must contain the payoff vector $[\frac{v(\bar{n})}{n},\ldots,\frac{v(\bar{n})}{n}]$. The payoff to a

coalition of k players is then $\frac{k}{n}$ $v(\bar{n})$, and this must exceed $v(\bar{k})$, for $k=1,...,n-1$.□

We begin with a classification of small games.

Proposition 5

For the class of symmetric superadditive games, in order to generate a non-empty core

(a) 1- and 2-person games do not require any unions;

(b) 3- and 4-person games require at most one union;

(c) 5- and 6-person games require at most two unions at one level.

Proof

(a) 1- and 2-person superadditive games are always proper.

(b) Obvious for 3-person games. If a 4-person game is not proper then the 3-person subgames are, for $v(\bar{4}) \geq 4v(\bar{1})$ and $v(\bar{4}) \geq 2v(\bar{2})$ by superadditivity, so if $v(\bar{4}) < \frac{4}{3} v(\bar{3})$ then $v(\bar{3}) \geq \frac{3}{2} v(\bar{2})$.

(c) If neither the 5-person game nor the 4-person subgames are proper then the 3-person subgames are from (b) above,

so form a 3-person union and a 2-person union. If the 6-person game is not proper, claim that either the 5- or the 4- person subgames are, so form a 5-person union in the first case and a 4-person union plus a 2-person union in the second case. To establish the claim, note that superadditivity implies $v(\bar{6}) \geq \frac{6}{k} v(\bar{k})$ for $k=1,2,3$. If the 6-person game is not proper then either $v(\bar{6}) < \frac{6}{4} v(\bar{4})$ or $v(\bar{6}) < \frac{6}{5} v(\bar{5})$ or both. The first possibility implies $v(\bar{4}) > \frac{4}{3} v(\bar{3})$ so that the 4-person subgames are proper. The second possibility implies $v(\bar{5}) \geq \frac{5}{2} v(\bar{2})$ and $v(\bar{5}) \geq \frac{5}{3} v(\bar{3})$. If also $v(\bar{5}) \geq \frac{5}{4} v(\bar{4})$ then 5-person subgames are proper, but otherwise we have $v(\bar{6}) < \frac{6}{4} v(\bar{4})$ which we have just shown implies that 4-person subgames are proper. This establishes the claim. □

For an arbitrary superadditive game a 4-person subgame is the largest one for which a non-empty core can be guaranteed within one further level of unions; for a symmetric superadditive game proposition 5(c) guarantees this for a 6-person subgame. Although

this increases by half the size of game that can be treated with a specified number of levels of unions, it will typically decrease by at most one the number of levels required. For example, a 1000-person symmetric game requires 8 instead of 7 levels. For large games the following lemma offers a much more significant reduction in the complexity of the union structure required.

Lemma 4

In any n-person symmetric superadditive game there is always a proper subgame containing at least $\left[\frac{n}{2}\right]+1$ players (where $\left[x\right]$ denotes the integer part of x).

Proof

The proof is by induction, taking even-person and odd-person games in turn. The proposition evidently holds for 1- and 2-person games. To establish the induction for even-person games is easy: if it holds for 2n-person games then evidently it holds for 2n+1-person games since the subgames in question are the same. To establish the induction for odd-person games we proceed by contradiction. Suppose that the proposition holds for 2n-1-person

games but not for 2n-person games. Then at least one of the n, n+1,..., 2n-1 person subgames is proper but none of the n+1, n+2,...,2n-person subgames are, so that n-person subgames must be proper. Now if 2n-person games are not proper then because none of the n+1, n+2,..., 2n-person games are proper, we can find a sequence $2n+k_o > k_1 > k_2 > \ldots > k_p$, with $k_p < n+1$, such that $v(\bar{k}_1) < \frac{k_i}{k_{i+1}} v(\bar{k}_{i+1})$ for $i=0,1,\ldots,p-1$. This implies $v(\overline{2n}) < \frac{2n}{k} v(\bar{k}_p)$. But $v(\overline{2n}) \geq 2v(\bar{n})$ by superadditivity and $v(\bar{n}) \geq \frac{n}{1} v(\bar{k})$ for $k=1,2,\ldots,n$ since n-person subgames are proper. Consequently $v(\overline{2n}) \geq \frac{2n}{k_p} v(\bar{k}_p)$, a contradiction. It follows that if the proposition holds for 2n-1 person games then it also holds for 2n-person games, which completes the induction for odd-person games and establishes the lemma. □

Corollary

A superadditive symmetric n-person game may be reduced to a superadditive non-symmetric m-person game, where

$$2^m - 2 < n \leq 2^{m+1} - 2,$$

by forming at most m proper unions at one level.

Proof

We proceed by induction on m. Clearly the corollary is true for m=1. Assume it to be true for arbitrary m-1, so that any symmetric game not exceeding 2^m-2 players may be reduced to an m-1 person game. Consider games with up to $2^{m+1}-2$ players. By lemma 4 we can form a proper union such that the remaining players number less than or equal to 2^m-2. But by assumption these players can then be reduced to at most n-1 proper unions, hence the original game is reduced to at most m players consisting of proper unions. □

The maximal subgame resulting from the application of this corollary is not necessarily proper, but we may form a bisecting ultimate union structure on it. Applying propositions 2a, 3 and 4(b) we have immediately

Proposition 6

An empty core may be generated for any n-person symmetric superadditive game by forming at most 2m-2 unions at h+1 levels,

where

$$2^m-2 < n \leq 2^{m+1}-2$$

and $2^h < m \leq 2^{h+1}$.

We may thus calculate that a symmetric 1000-person game may be reduced to a 9-person game by forming 9 unions at one level. It requires at most 7 further unions at 3 further levels to generate a non-empty core for the original game, giving a total of 16 unions at 4 levels. This is in contrast to an upper bound of 998 unions at 9 levels for an arbitrary non-symmetric game.

A Postlewaite has shown (in private correspondence) that one union suffices to generate a non-empty core for a non-symmetric superadditive 4-person game, so that 5(a) and (b) hold for all superadditive games, but it is not known whether 5(c) does. Postlewaite has also shown that a non-empty core may be generated for any n-person superadditive game (n even) by appropriately choosing n/2 players for a first union and subsequently adding players one at a time to form a centred union structure. This requires only n/2 unions but at n/2 levels.

... ems likely that tighter bounds can be obtained than those given in the present paper. It would also be useful to obtain analogues of lemma 4 for different classes of games.

VIII FURTHER DISCUSSION

The notion of union structure developed above seems to be intuitively reasonable. In any bargaining situation some or all of the ultimate participants form into unions or other organisations. Delegates of these unions in turn form more or less binding alliances and choose spokesmen, and so on. The strength of a union S is in two respects. First, by establishing its right to block it guarantees its members at least the amount v(S). Second it protects its members against "exploitation" by non-members by the rule of solidarity.

If players in a game can form unions, then we have seen that it is no longer necessary for the final imputation to be in the original core of that game. Moreover it is possible that a process of disaggregation might take place. If an initial player really represents a union of smaller players and that union breaks up then the core may shrink. In this case the

original core may overstate the range of likely outcomes. The original core thus loses some of its former significance. Rather one must look to the whole set of cores which might be generated by union structures.

These considerations suggest that the choice of players in specifying a game is a rather arbitrary one. The players themselves may modify the game by their own actions. One is, therefore, led to consider which union structures are the most likely. In Example 1 it seems natural for the two small men to form a union; in Example 2 a conflict situation seems unavoidable.

In general terms, a union will form if the benefits of doing so exceed the costs. The benefits will depend upon the rules of allocation within and between unions. What constitutes a "reasonable rule" is much-debated and no doubt depends upon the alternative opportunities available to the players. The costs are of two kinds: the opportunity costs of committing oneself to a particular coalition, thereby rejecting other possibilities, and the transactions costs of obtaining

information, negotiating terms, forming the union, and policing the agreements. In particular, it is important to ask whether the unions required to generate an empty core are those which are likely to be formed by the participants. Such issues remain to be explored, but we indicate here one possible line of approach if an allocation rule can be agreed upon and enforced (for example, as a result of government legislation).

For each union structure, the payoff vector corresponding to a particular rule (eg. the Shapley value or the nucleolus) may be defined in a natural way via the modified subgames. Say that a union structure is "locally stable" if there is no incentive for a single existing union to disband because at least one member would be better off without the union nor any incentive for a single additional union to form because all potential members would be better off with the new union.

Consider the class of zero-one normalised three-person games under the rule of the Shapley value function. There are 3 possible union structures (each with a single essential union)

in addition to $\mathcal{U}_o$. Since no union structure can be transformed into another without going via $\mathcal{U}_o$, there is no possibility of "cycling". It follows that there must exist at least one locally stable union structure. In fact, it may be shown by enumeration that each of the four possible union structures is locally stable for some game. (More precisely, $\mathcal{U}_o$ is locally stable if the characteristic value of any two-person coalition does not exceed 1/5, and the three other union structures are locally stable if the characteristic value of any two-person coalition is not less than 3/5). In example 1 above, the union of the two small men is the only locally stable union structure, while in example 2 all three two-person unions are locally stable. In order to explore notions of global stability, including the possibility of retaliation, it seems natural to define the bargaining set and kernel under union structures. These aspects are beyond the scope of the present paper.

FOOTNOTES

1 We are indebted for this interpretation to M. Keane, who in turn attributes it to M. Maschler. There are many other interpretations of the game eg. one bolt and two nuts.

2 This game is in fact formally equivalent to that with two big men and one small man.

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